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On Burnside's Theorem*

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Let χ be an irreducible character of the finite group G. Apply χ to the entries in the multiplication table of G. Call the resulting matrix M. Let S be a subset of G. If $\{A(g) : g \in G\}$ is any representation affording χ, then $\{A(g) : g \in S\}$ is linearly independent if and only if the rows of M corresponding to S are linearly independent.

Let G be a finite group. Let χ be an irreducible complex character of G, with $\chi(id) = n$. The title theorem of Burnside is this: If $\alpha = \{A(g) : g \in G\}$ is a representation of G which affords χ, then the elements of α span $C_{n,n}$, the vector space of all complex n by n matrices. In view of the difficulty in obtaining explicit irreducible representations of groups, the following would appear to be a natural question: Given only χ and the multiplication table for G, can one choose $g_1, g_2, \ldots, g_{n^2}$ from G so that if α were explicitly known, $\{A(g_i): 1 \leqslant i \leqslant n^2\}$ would be a basis of $C_{n,n}$? The answer to this question has application, for example, in choosing "induced bases" for certain symmetry classes of tensors [2]. Burnside himself came close to the following answer [1, Sect. 230].

THEOREM. *Apply χ to the entries in the multiplication table for G. Call the resulting matrix M. Let S be a subset of G. Then $\{A(g): g \in S\}$ is linearly independent if and only if the rows of M corresponding to S are linearly independent.*

Proof. Consider the element

$$t(G, \chi) = (n/o(G)) \sum_{g \in G} \chi(g)g$$

of the complex group algebra CG. Then $t(G, \chi)$ is the central indempotent which generates the minimal two-sided ideal to which χ corresponds. Indeed, considering $g \in G$ as a linear operator on CG via left multiplication (i.e., the left regular representation), the restriction of g to the ideal generated by $t(G, \chi)$ is

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equivalent to the direct sum of $A(g)$ with itself n times. Thus $\{A(g): g \in S\}$ is linearly independent if and only if $\{gt(G, \chi): g \in S\}$ is linearly independent. But, $gt(G, \chi) = t(G, \chi) g$, for all $g \in G$. Therefore, considering $t(G, \chi)$ itself as an operator on CG, we see that $\{A(g): g \in S\}$ is linearly independent if and only if $\{t(G, \chi) g : g \in S\}$ is a linearly independent set in the range of $t(G, \chi)$. With respect to the basis $\{g \in G\}$ of CG, the matrix representation R of $t(G, \chi)$ is easily seen to have a g_i, g_j entry equal to $n\chi(g_i g_j^{-1})/o(G)$. Factoring out the common multiple $n/o(G)$ and permuting the columns of R appropriately, one obtains M.

Remark. The matrix R in the proof is Hermitian indempotent and therefore positive semidefinite. As Christopher Morgan pointed out to me, it is more suited to computations than is M; only principal submatrices need be considered.

References

1. W. Burnside, "Theory of Groups of Finite Order," 2nd ed., Cambridge Univ. Press, 1911. (Reprinted by Dover, New York, 1955).
2. R. Merris, The structure of higher degree symmetry classes of tensors, *J. Res. (Nat. Bur. Standards B)* **80** (1976), 259–264.

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